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A Fuzzy Programming Approach to solve Over Determined and Inconsistent System of Equations arising in a CAT SCAN

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ABSTRACT

Image reconstruction from tomographic data is an increasingly important area of applied linear algebra particularly for medical diagnosis. The fundamental mathematical problem here is finding solution to large system of linear equations of the form $\sum_j a_{ij} x_j = b_i$. In transmission tomography the unknown entries x_j represent intensity levels of beam attenuation. Because of the inherent modelling and experimental errors such equations need not be consistent due to which getting any solution, leave alone a unique solution is ruled out. In such a situation approximate solutions especially those satisfying as many equations as possible to a certain degree of acceptance would be desirable. The proposed method deals with inconsistency of the system of equations by using fuzzy equations instead of crisp equations. Experimental as well as modeling errors are also accommodated in such a modification. Solution methodology is intended for obtaining solutions of the fuzzy equations with the help of Fuzzy Programming techniques. It gives various possible values for the pixel densities x_j and imaging these as functions of the spatial locations will give pictures of the material within the body. Thus different pictures of the object can be reconstructed. So this method has the distinct advantage of providing different sets of scanned reports, which will be helpful in detecting the symptoms of certain diseases.

INTRODUCTION

Computer Aided Tomography (CAT) is a diagnostic tool which plays an important role in therapy. The underlying principle of CAT technology is that the object being imaged is modeled as a discrete set of points using the fact that the intensity of an X-ray passing through a body is decreased by an amount proportional to the mass of the tissue and the distance the X-ray travels. Materials such as bone and teeth block more of the X-rays resulting in a lower signal compared to soft tissues and fat. CAT can accurately measure the amount of X-ray absorption of tissues thus indicating the nature of tissues being studied.

CAT scanners provide a picture of the inside of the body by making use of a coordinate grid and evaluating the mass of objects measured indirectly with X- rays. The source and detector assemblies are translated at a particular angle to acquire a single view. A complete CAT scan requires 300 to 1000 views taken at rotational increments 0.3 to1 accomplished by mounting the X-ray source and detector on a rotating gantry that surrounds the patient.

Given a set of views there are four main approaches for calculating cross sectional image. These are called Reconstruction Algorithms. The first uses a very large number of linear equations to reconstruct the picture. The second method uses iterative techniques to calculate the final image in small steps through Simultaneous Iterative Reconstruction Techniques (SIRT). In both cases these equations need not be consistent. The suggested methodology can handle the set of inconsistent equations in such away as to provide various possible solutions. The other two approaches are based on Digital Signal Processing.

The paper is organized as follows. In Section 1 an excerpt from “Elementary Linear Algebra” by Howard Anton and Chris Rorres [1] is reproduced to explain how a CAT scanner works and gives rise to a set of equations. Section 2 presents the relevance of an equivalent fuzzy system and gives a mathematical model together with a methodology for getting solutions to fuzzy equations. A numerical example to illustrate the methodology developed is given in Section 3.Conclusions are given in Section 4.

SECTION I

The working of CAT scanner: The basic problem of computed tomography is to reconstruct an image of a cross section of the human body using data collected from many individual beams of X rays that are passed through the cross section.

These data are processed by a computer and the computed cross section is displayed on a video monitor. Such a system is known as a CAT scanner, for computer Aided Tomography scanner. The first commercial system of computed tomography for medical use was developed in 1971 by G. N. Houndsfield of EMI Ltd. in England. In 1979 Houndsfield and A. M. Geomarck were awarded the Nobel prize for their pioneering work in the field. In this section how a tomography requires the solution of a large linear system of equations is explained. Existing methods use certain algorithms called algebraic reconstruction techniques to solve these

linear equations whose solution yields the cross sections in digital form.

Unlike conventional X-ray pictures framed by X rays that are projected perpendicular to the plane of the picture, tomographs are constructed from thousands of individual hairline X-ray beams that lie in the plane of the cross sections. The intersection of the X ray beams are measured by an X-ray detector, and these measurements are relayed to a computer when they are processed. There are two possible modes of scanning the cross section, parallel mode and fan beam mode. How the cross section is reconstructed from the individual beam measurements is explained as follows. The field of view in which the cross section is situated has been divided into many square pixels (picture elements) numbered 1 through n. X- ray density of each pixel is determined and they are reproduced on a video monitor with each pixel shaded a level of gray proportional to its X-ray density. Because different tissues within the human body have different X-ray densities, the video display clearly distinguishes the various tissues and organs within the cross section. The photons constituting the X-ray beam are absorbed by the tissue within the pixel at the rate proportional to the X-ray density of the tissue. The X-ray density of the j^{th} pixel is denoted by x_j and is defined by

$$x_j = \ln\left(\frac{\text{number of photons entering } j^{th} \text{ pixel}}{\text{number of photons leaving } j^{th} \text{ pixel}}\right)$$

= - ln (fraction of photons that pass through the j^{th} pixel without being absorbed)

If the X-ray beam passes through an entire row of pixels then the number of photons leaving one pixel is equal to the number of photons entering the next pixel in the row. If the pixels are numbered from 1 to n then by the additive property of the logarithmic function

$$x_1 + \ldots\ldots + x_n = \ln\left(\frac{\text{number of photons entering the first pixel}}{\text{number of photons leaving the nth pixel}}\right)$$

i.e., to get the total X-ray density of a row of pixels, we simply sum the individual pixel densities.

Now the beam density of the i^{th} beam of a scan is denoted by b_i

$$b_i = \ln\left(\frac{\begin{array}{c}\text{number of photons of the } i^{th} \text{ beam}\\ \text{entering the detector without the cross}\\ \text{section in the field of view}\end{array}}{\begin{array}{c}\text{number of photons of the } i^{th} \text{ beam}\\ \text{entering the detector with the cross}\\ \text{section in the field of view}\end{array}}\right)$$

= - ln (fraction of photons of the i^{th} beam that pass through the cross section without being absorbed)

Numerator of the first expression for b_i is obtained by performing a calibration scan without the cross section in the field of view and denominator is obtained by performing a scan with the cross section in the field of view. These values are stored for further processing.

For each beam that passes squarely through a row of pixels.

$$\left(\begin{array}{l}\text{fraction of photons of the beam that}\\ \text{pass through the row of pixels}\\ \text{without being absorbed}\end{array}\right)$$

$$= \left(\begin{array}{l}\text{fraction of photons of the beam that}\\ \text{pass through the cross section without}\\ \text{being absorbed}\end{array}\right)$$

ie $x_1 + \ldots\ldots\ldots.. + x_n = b_i$

In this equation b_i is known from the clinical and calibration measurements and $x_1, \ldots\ldots x_n$ are unknown pixel densities that must be determined.

More generally, if the i^{th} beam passes squarely through a row or column of pixels with numbers $j_1, j_2, \ldots j_i$ then we have

$x_{j_1} + x_{j_2} + \ldots\ldots + x_{j_i} = b_i$

If we set $a_{ij} = \begin{cases} 1 & \text{if } j = j_1, j_2, \ldots.,.j_i \\ 0 & \text{otherwise} \end{cases}$.

Then we may write this equation as

$a_{i1}x_1 + a_{i2}x_2 + \ldots\ldots + a_{iN}x_N = b_i$ which is known as the i^{th} beam equation.

SECTION II
Mathematical Model

From the excerpt given in section 1 it is clear that a system of m beam equations in n pixel densities can be written as $\sum_j a_{ij}x_j = b_i$, $i = 1\ldots m$. Here x_j represents intensity levels of beam attenuation. The attenuation function is discretized, by imagining the body to consist of finitely many squares or pixels within which the function has a constant but unknown value. Thus x_j is the X- ray density of the j^{th} pixel. The beam is sent through the body along various lines L_i and both initial and final beam strengths are measured. Then a_{ij}'s may be taken as either 1 or 0 according as the line L_i intersects the jth pixel or not. The entries b_i are typically counts of detected photons. It may be noted that in the set of equations $\sum_j a_{ij}x_j = b_i$ in pixel densities x_j, b_i as well as x_j are nonnegative. Because of the inherent modeling and experimental errors, these equations are usually inconsistent as well as overdetermined. So that the system need not have an exact mathematical solution. Hence the aim here is to obtain a somewhat acceptable solution to this system.. for which a fuzzy equality is considered rather than a

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strict equality which can admit solutions with varying degree of satisfaction. This is achieved by introducing fuzzy equations to solve for pixel densities. These equations are solved by fuzzy programming. To proceed in this direction a brief introduction of fuzzy sets and fuzzy programming is required.

Basic definitions

1. If X is a collection of objects denoted by x , then a fuzzy set $\tilde{A}$ is a set of ordered pairs

$\tilde{A}$ = { (x, $\mu_A(x)$) / $x \in X$ }. Here $\mu_A(x)$ is called the membership function or grade of membership of x in $\tilde{A}$ which maps X in to the interval [0,1]. If the set of membership values contains only 0 and 1 then μ is identical with the characteristic function and $\tilde{A}$ is then a non fuzzy (crisp) set. Elements with a zero degree membership are normally not listed. Hence a fuzzy set is a generalization of a classical set and the membership function a generalization of characteristic function.

Example $\tilde{A}$ = integers close to 10

Then $\tilde{A}$ may be

={0.1/7, 0.5/8, 0.8/9, 1/10, 0.8/11, 0.5/12,0.1/13}

2. The support of a fuzzy set $\tilde{A}$,

S($\tilde{A}$) = {x/ $x \in X$ $\mu_A(x) > 0$}

3. The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called α level or α cut set. i.e,

$$A_\alpha = \{x / x \in X, \mu_A(x) \geq \alpha\}$$

$$A_\alpha' = \{x / x \in X, \mu_A(x) > \alpha\}$$

are called as α cut set and strong α cut set.

Let the corresponding system of fuzzy equations be $g_i = \sum_j a_{ij} x_j \approx b_i$ where ≈ denotes fuzzy equality. This means that g_i is around or in the vicinity of the number b_i. To be specific, let the range of g_i be taken as $(b_i - rq_i, b_i + r q_i')$ for some suitable q_i and q_i' with membership functions as

$$\mu_i(g_i) = \begin{cases} \dfrac{g_i - (b_i - q_i)}{q_i} & if \quad b_i - q_i \leq g_i \leq b_i \\ \dfrac{(b_i + q_i') - g_i}{q_i'} & if \quad b_i \leq g_i \leq b_i + q_i' \end{cases}$$

For each fuzzy equation $g_i = \sum_j a_{ij} x_j \approx b_i$ consider the (1- r) cuts sets $\{g_i / \mu_i(g_i) \geq (1-r)\}$ so that these equations are satisfied to the degree (1-r) = p where p is called the degree of satisfaction. Thus the elements of the α level set of ith constraint are to satisfy

$$\frac{g_i - (b_i - q_i)}{q_i} \geq (1-r) = p$$

$$\frac{(b_i + q_i') - g_i}{q_i'} \leq (1-r) = p$$

Each fuzzy equation is thereby converted into a set of two crisp inequalities of the form

$$\sum a_{ij} x_j \geq b_i - q_i + p\, q_i, \qquad i = 1, \ldots, m$$

$$\sum a_{ij} x_j \leq b_i + q_i' - p\, q_i', \quad i = 1, \ldots, m \quad \text{where } 0 \leq p \leq 1. \qquad (1)$$

Solution x_j j = 1,2,…,m of the above 2m inequalities provide g_i values with in the range ($b_i - q_i$, $b_i + q_i$).

Solution Methodology

Slack and surplus variables are introduced to change the inequalities into equations. To get a solution to this system of linear equations phase I of the II phase technique in Linear programming is used. Using parametric programming the range within which this solution remains optimal is determined.

Stage I

Now the prolem is to solve a parametric programming problem. First it is solved for r = 0 and analysis done for r in the interval [0,1]. Here predetermined linear variations in the direction of the vector $\hat{H} = (-q_1, q_1', -q_2, q_2', \ldots\ldots\ldots, -q_m, q_m')^T$ is considered. Let B be the initial basis matrix and X_B the set of basic variables with X^0 as the solution for r = 0. As r changes, the feasibility of the solution is affected. The optimal basis matrix B for the program is also a basis matrix for the modified program. (i.e.,when H is changed to $H + r\hat{H}$).

if

$$X_B = B^{-1}(H + r\hat{H}) = B^{-1}H + rB^{-1}\hat{H}$$

$$= X^0 + rB^{-1}\hat{H} \geq 0$$

If $X^0 + rB^{-1}\hat{H} \geq 0$ then this basic solution is feasible and also optimal since the relative cost factors are nonnegative.

For this to be nonnegative

$$r \geq \frac{-(X^0)_i}{\eta_i} > 0$$

if $\eta_i = (B^{-1}\hat{H})_i > 0$

$$r \le \frac{-(X^0)_i}{\eta_i} > 0 \quad \text{if } \eta_i < 0$$

where η_i is the i^{th} component of $B^{-1}\hat{H}$ and $(X^0)_i$ is the value of the i^{th} basic variable.

Since $r \ge 0$, let the range obtained be $(0, r_1)$

where $r_1 = Min\left\{\frac{-(X^0)_i}{\eta_i}, \quad \eta_i < 0\right\}$

This means that if $r > r_1$ some components of X^0 are negative and hence x^0 is not primal feasible but dual feasible for the modified program. Consequently dual simplex method can be applied to obtain new optimal solution to the modified program. Repeated application of this procedure will yield solutions which are valid in the respective intervals of r.

If optimal solution does not exist for the parametric program for r = 0 then it is solved for r = 1 and the range of r for which the basis matrix remains the same is obtained.

Here $r_1' = Max\left\{\frac{-(X^0)_i}{\eta_i}, \quad \eta_i > 0\right\}$ and the relevant interval is $(1 - r', 1)$

Also this procedure is repeated for getting optimal solutions for various values of r.

Let $X^0(r), X^1(r), \ldots, X^{s-1}(r)$ be the respective solutions in the intervals $r_0 \le r \le r_s$ where $r_0 \ge 0$ and $r_s \le 1$. For each solution $x^i(r)$ the degree of satisfaction p for any constraint is at least (1- r). So the various solutions in terms of the degree of satisfaction p are

$X^0(p)$ for $p_1 \le p \le p_o$
$X^1(p)$ for $p_2 \le p \le p_1$
:
:
$X^{s-1}(p)$ for $p_s \le p \le p_{s-1}$ where $p_i = 1 - r_i$

The maximum degree of satisfaction is p_0 with corresponding solution $X^0(p_0)$.
Then the maximum value, of the degree of satisfaction p, of all the constraints is α with p being the minimum degree with which each constraint is satisfied.

Stage II. (Moving towards better solution)
A second stage problem to improve (if possible) the solutions obtained in stage I in the range $\eta \le p \le \alpha$ is stated as

Max $\sum_i \beta_i$

subject to $\alpha \le \beta_i \le 1, \quad i = 1,\ldots,m$

$\beta_i \le \mu_i(g_i)$

$x_j \ge 0.$

where β_i represents the degree of satisfaction of the i^{th} constraint.

If in the optimal solution of it any of the β_i exceeds p then an improved solution, in the sense that it attains a higher membership value to at least one constraint, is obtained. Consequently with this new solution some of the equations will be satisfied in a better way. These solutions for each p in $\eta \le p \le \alpha$ are non dominated in nature because no solution is inferior to other solutions with respect to degree of satisfaction . Imaging these solutions will provide different pictures of the object.

SECTION III

Numerical Example: Consider the following set of crisp overdetermined equations

$g_i = \sum_j a_{ij} x_j = b_i \quad i = 1,\ldots,4 \quad j = 1,\ldots,3$ given by

$x_1 + x_2 + x_3 = 5$;
$2x_1 + x_2 + x_3 = 6$;
$x_1 + 2x_2 + 2x_3 = 6$;
$x_1 + x_2 + 2x_3 = 4$
which are inconsistent.

Instead of crisp equations let fuzzy equations be used with membership values μ_i for g_i
i =1,2,3,4 given by

$$\mu_1(g_1) = \begin{cases} \frac{g_1 - 3}{2} & if\ 3 \le g_1 \le 5 \\ \frac{7 - g_1}{2} & if\ 5 \le g_1 \le 7 \end{cases}$$

$$\mu_i(g_i) = \begin{cases} \frac{g_i - 4}{2} & if\ 4 \le g_i \le 6 \\ \frac{8 - g_i}{2} & if\ 6 \le g_i \le 8 \end{cases} \quad i = 2,3$$

$$\mu_4(g_4) = \begin{cases} \frac{g_4 - 4}{2} & if\ 2 \le g_4 \le 4 \\ \frac{6 - g_4}{2} & if\ 4 \le g_4 \le 6 \end{cases}$$

The transformed optimization problem is

Min $x_5 + x_8 + x_{11} + x_{14}$

such that
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 + x_5 &= 3+2p \\ x_1 + x_2 + x_3 + x_6 &= 7-2p \\ 2x_1 + x_2 + x_3 - x_7 + x_8 &= 4+2p \\ 2x_1 + x_2 + x_3 + x_9 &= 8-2p \end{aligned}$$

$x_1 + 2x_2 + 2x_3 - x_{10} + x_{11} = 4+2p$
$x_1 + 2x_2 + 2x_3 + x_{12} = 8-2p$
$x_1 + x_2 + 2x_3 - x_{13} + x_{14} = 2+2p$
$x_1 + x_2 + 2x_3 + x_{15} = 6-2p$
$x_j \geq 0$, $j = 1,2, \ldots,12$;
$0 \leq p \leq 1$

Solving for $p = 0$,
$x_1 = 8/3$, $x_2 = 2$, $x_3 = 2/3$ $x_4 = 7/3$, $x_6 = 5/3$, $x_7 = 4$, $x_{10} = 4$, $x_{13} = 4$.

Performing Parametric Programming analysis solutions for various values of p in the range $0 \leq p \leq 0.7$ are

$x_1 = \frac{8}{3} - \frac{2}{3}p$,
$x_2 = 2$,
$x_3 = \frac{2}{3} - \frac{2}{3}p$,
$x_4 = \frac{7}{3} - \frac{10}{3}p$,
$x_6 = \frac{5}{3} - \frac{2}{3}p$,
$x_7 = 4 - 4p$,
$x_{10} = 4 - 4p$
$x_{13} = 4-4p$
and $x_j = 0$, $j = 5, 8, 9, 11, 12, 14, 15$.

Stage I				Stage II				
Degree of satisfaction p	Solution			Degrees of satisfaction	Solution			Z
	x_1	x_2	x_3		x_1	x_2	x_3	
0.7	2.2	2	0.2	$\beta_1 = 0.7$ $\beta_2 = 0.7$ $\beta_3 = 0.7$ $\beta_4 = 0.7$	2.2	2.2	0	2.8
0.65	2.23	2	0.23	$\beta_1 = 0.65$ $\beta_2 = 0.65$ $\beta_3 = 0.9$ $\beta_4 = 0.85$	2.4	1.9	0	3.05
0.6	2.27	2	0.27	$\beta_1 = 0.6$ $\beta_2 = 0.7$ $\beta_3 = 1$ $\beta_4 = 0.9$	2.4	1.8	0	3.2
0.5	2.33	2	0.33	$\beta_1 = 0.5$ $\beta_2 = 1$ $\beta_3 = 1$ $\beta_4 = 1$	2	2	0	3.5
0.4	2.4	2	0.4	$\beta_1 = 0.5$ $\beta_2 = 1$ $\beta_3 = 1$ $\beta_4 = 1$	2	2	0	3.5

The solution for the maximum degree of satisfaction p = 0.7 of all the equations gives x_1=2.2, x_2 = 2, x_3 = 0.2 so that the equations are satisfied as

$x_1 + x_2 + x_3 = 4.4$
$2x_1 + x_2 + x_3 = 6.6$
$x_1 + 2x_2 + 2x_3 = 6.6$.
$x_1 + x_2 + 2x_3 = 4.6$

Improvement of this solution through second stage is not possible since each equation in this case is satisfied with the same degree 0.7

Other equally good solutions for different p values can be obtained through the introduction of stage II linear programming problem. In the case of p = 0.5 it is as given below.

Max $\beta_1 + \beta_2 + \beta_3 + \beta_4$
Subject to $0.5 \leq \beta_i, \leq 1, \quad i = 1,\ldots,4$
$x_1+x_2+x_3 - 2\beta_1 \geq 3$
$x_1+x_2+x_3 + 2\beta_1 \leq 7$
$2x_1+x_2+x_3 - 2\beta_2 \geq 4$
$2x_1+x_2+x3 + 2\beta_2 \leq 8$
$x_1+2x_2+2x_3 - 2\beta_3 \geq 4$
$x_1+2x_2+2x_3 + 2\beta_3 \leq 8$
$x_1+ x_2+ 2x_3 - 2\beta_4 \geq 2$
$x_1+ x_2+ 2x_3 + 2\beta_4 \leq 6$
$x_j \geq 0$, $j = 1,\ldots,3$

Optimum solution is x_1 = 2, x_2 = 2, x_3 = 0 satisfying the following equations with degrees of satisfaction β_1 = 0.5, β_2 = 1, β_3 = 1, β_4= 1

$x_1 + x_2 + x_3 = 4$
$2x_1 + x_2 + x_3 = 6$
$x_1 + 2x_2 + 2x_3 = 6$
$x_1 + x_2 + 2x_3 = 4$

This is definitely a better solution than the one obtained in stage I for p = 0.5 For p = 0.6 the solution is x_1 = 34/15 ,x_2 = 34/15, x_3 = 0 and its improved form is x_1 = 2.4, x_2 = 1.8, x_3 = 0 with degrees of satisfaction respectively for the constraints as 0.6, 0.7, 1, 0.9. The last column gives Z , the total of the degrees of satisfaction for all the constraints in the stage II problem.. It is worth mentioning that the maximum value Z= 3.5 is attained by all values of p $\in$(0.5, 1) Various possible solutions to stage I and stage II problems with respective degrees of satisfaction are depicted in the following table.

SECTION IV

Conclusions: This method has the distinct advantage of providing a set of solutions for various degrees of satisfaction of the equations, and the solution for the maximum degree of satisfaction reproduces the best cross sectional view in digital form. It is also possible to obtain more number of cross sectional views even though with a lesser degree of satisfaction for a few constraints but with better degree of satisfaction for others. This method is able to suggest a number of non dominated solutions in terms of the degree of satisfaction, through which many scanned reports may be formed. These different sets of scanned reports may be useful in detecting symptoms of certain diseases.

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